Equivalence of Continuum and Discrete Methods of Shape Design Sensitivity Analysis

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This paper presents results on the study of equivalence of discrete-discrete (D-D) and continuum-discrete (C-D) methods of shape design sensitivity analysis. For the D-D method, discrete design sensitivity expressions are obtained by taking analytical derivatives of the finite-element matrix equation with respect to shape design variables. For the C-D method, derivatives of the continuum equilibrium equation are taken using the material-derivative idea to obtain the design sensitivity expressions in continuous setting. In both D-D and C-D methods, the design sensitivity expressions are evaluated using the analysis results of the finite-element method. It is shown in this paper that, for the shape design variable, they are not equivalent. For the equivalence study, the same discretization method (shape function) used for the finite-element model is used to evaluate the continuum design sensitivity expressions. Also, a consistent method is employed for the movement of finite-element grid points for shape design change in the D-D method and the discretization of the design velocity field of the C-D method. Moreover, it is shown that the result of the C-D method can detect inaccuracy of the result of finite-element analysis. The reason is because the continuum design sensitivity expressions must provide accurate design sensitivity information when accurate finite-element analysis results are used to evaluate these expressions.

I. Introduction

SUBSTANTIAL literature has emerged in the field of sizing and shape design sensitivity analysis (DSA) of structural systems and machine components (Refs. 1–7 and references cited therein) over the last 25 years. DSA has recently emerged as a much needed design tool, not only from the standpoint of the role of design sensitivity information in optimization, but also from the standpoint of using design sensitivity information in a computer-aided engineering environment for interactive design. Also, design sensitivity information can be used for approximate analysis and analytical model improvement.

Developments in the DSA method have been made using two fundamentally different approaches. In the first approach, a discretized structural finite-element model is used to carry out DSA. There are three different methods in the discrete DSA approach: finite difference, semianalytical, and analytical.

The finite-difference method is a popular one due to its simplicity, but a serious shortcoming of the method is the uncertainty in the choice of a perturbation step size of the design variables. Moreover, for the shape design problems, the method is quite costly.[4] In the semianalytical method, the derivatives of the stiffness matrix are computed by finite differences,[8-10] whereas in the analytical method, the derivatives are obtained analytically. But for elements having bending stiffness, such as beams and plates, the stiffness matrix is a nonlinear function of the cross-sectional sizes and the stiffness matrix derivatives are not easily evaluted.[11] Moreover, in the case of shape design variables, computation of analytical derivatives of the stiffness matrix is quite costly.[4] Consequently, the semianalytical method is a popular choice in the discrete-shape DSA approach[9,10] However, recently, Barthelemy and Haftka[12] showed that the semianalytical method can have serious accuracy problems for shape design variables in structures modeled by beam, plate, truss, frame, and solid elements. They found that accuracy problems occur even for a simple cantilever beam. This accuracy problem of the semianalytical method has been confirmed by Pedersen et al.[13]

In the second approach, a continuum model of the structure is used to carry out the DSA. For shape design variables, the material-derivative concept of continuum mechanics is used to relate variations in structural shape to measures of structural performance.[14-18] Using the continuum DSA approach, expressions for shape design sensitivity are obtained in the form of integrals with integrands written in terms of natural physical quantities such as displacements, stresses, strains, and domain shape changes. For the shape design sensitivity, alternate but analytically equivalent formulations can be obtained in the form of boundary integrals and domain integrals.[7,16,18] If exact solutions of the continuum equilibrium equations are used to evaluate these continuum design sensitivity expressions, the method is called the continuum-continuum (C-C) method. On the other hand, if the analysis results of the finite-element or boundary-element methods are used to evaluate these terms, the method is called the continuum-discrete (C-D) method. The analytical method of the discrete design sensitivity analysis approach will be called the discrete-discrete (D-D) method.

The C-C method provides the exact design sensitivity of the exact model, whereas the C-D method provides an approximate design sensitivity of the exact model. On the other hand, the D-D method yields the exact design sensitivity of an approximate finite-element model and both the finite-difference and semianalytical methods yield approximate design sensitivities of an approximate finite-element model.

One question often asked is: "Are the D-D and C-D methods equivalent?" For this question, certain conditions have to be met. First, the same discretization (shape function) used for the finite-element analysis method must be used to evaluate the continuum design sensitivity results. Second, exact integrations (instead of numerical integrations) must be carried out

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Table 1 Results of equivalence study of D-D and C-D methods for truss

	Loading condition					
	p		f		qx/ℓ	q
	Shape function					
Design velocity field	Linear	Quadratic	Linear	Quadratic	Linear	Quadratic
Linear	Same	Same	Same	Same	Same	Same
Quadratic	Same	Same	Not	Same	Not	Not
Cubic	Same	Same	Not	Not	Not	Not

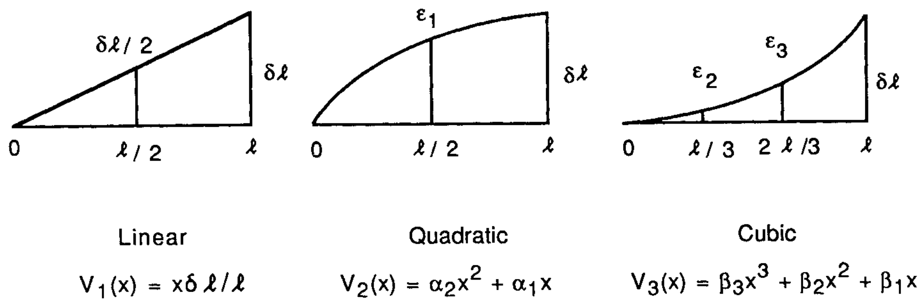

Fig. 1 Parameterizations of design velocy $V(x)$.

for all integrations used for generation of the stiffness matrix and evaluation of the continuum design sensitivity expressions. The third condition to satisfy is that the exact solutions (not a numerical solution) of the finite-element matrix equation and adjoint equation are used to compare the two methods. The fourth condition is that movement of the finite-element grid points for shape design change in the D-D method must be consistent with the parameterization method used for the design velocity field of the C-D method. For the sizing design variable, it is shown in Ref. 7 that the D-D and C-D methods are equivalent under the conditions given above using a beam structural component. It has also been argued that the D-D and C-D methods are equivalent for the shape design variable under the conditions given above.[19,20] One point to note is that these four conditions are not easy to satisfy; in many cases, numerical integrations are used and exact solutions of the finite-element matrix equations cannot be obtained. In this paper, equivalence study of the D-D and C-D methods for shape design variables is carried out under the conditions given above.

II. Design Sensitivity Analysis Methods

The D-D method starts with the finite-element equilibrium equations for a linear structural system as

$$K(b)z = F(b) \tag{1}$$

where $K(b)$ is the reduced global stiffness matrix, z the reduced displacement vector, $F(b)$ the external load vector, and b a design variable vector. Differentiating both sides of Eq. (1) with respect to b gives

$$K(b)\frac{\mathrm{d}z}{\mathrm{d}b} = -\frac{\partial}{\partial b}[K(b)\tilde{z}] + \frac{\partial F(b)}{\partial b} \tag{2}$$

where the tilde indicates a variable that is to be held constant for the process of partial differentiation. The derivatives of the displacement vector, $\mathrm{d}z/\mathrm{d}b$, can be computed by solving Eq. (2). This is known as the direct differentiation method. If derivatives of a general performance measure are needed, an adjoint variable method can be used.[7] Even though the direct differentiation and adjoint variable methods are different in computational efficiency depending on the situation, they are equivalent in accuracy as long as a consistent computational procedure is used for both methods. For the D-D method, the derivative of the stiffness matrix in Eq. (2) is obtained analytically. The D-D method is applicable to both sizing and shape design variables. For the shape design case, the design variables are positions of the finite-element grid points.

For the C-D method, the variational governing equation is obtained from the principle of virtual work as[7]

$$a_\Omega(z,\bar{z}) = \ell_\Omega(\bar{z}) \tag{3}$$

which must hold for all kinematically admissible virtual displacements $\bar{z}$. In Eq. (3), $a_\Omega(\cdot,\cdot)$ denotes the energy bilinear form, $\ell_\Omega(\cdot)$ denotes the load linear form, and Ω is the shape of the structure. Note that an approximate finite-element matrix Eq. (1), is obtained by applying the Galerkin method to the continuum equilibrium Eq. (3), for an approximate solution. Taking the material derivative of both sides of Eq. (3),[7,16,18]

$$a_\Omega(\dot{z},\bar{z}) = \ell'_V(\bar{z}) - a'_V(z,\bar{z}) \tag{4}$$

which must hold for all kinematically admissible virtual displacements $\bar{z}$. In Eq. (4), $\dot{z}$ denotes the material derivative of the displacement z and V is the design velocity field.[7,16,18] Expressions for $a'_V(z,\bar{z})$ and $\ell'_V(\bar{z})$ can be obtained for various structural components.[7,16,18] Equation (4) is a variational equation for the material derivative $\dot{z}$. If the C-D method is used, an approximate finite-element matrix equation can be used to obtain an approximate solution of Eq. (4). On the other hand, for the C-C method, the analytical solution of Eq. (3) is used in Eq. (4) to obtain the analytical solution of Eq. (4). Thus, the C-C method gives the exact design sensitivity of the exact model. As in the D-D method, if derivatives of a general performance measure are needed, an adjoint variable method can be used.[7,16,18]

In this paper, to carry out the equivalence study of the D-D and C-D methods, two simple structural components, a truss and a cantilever beam, are used. The shape DSA results of the D-D and C-D methods derived in the published literature are cited and used here without derivation.

III. Truss

In this section, shape design sensitivities of a simple truss with one end fixed are carried out using the D-D and C-D methods to study equivalence. The truss has a uniform cross-sectional area A and its length is ℓ. Three loading cases—a point load p, a uniformly distributed load f, and a linearly varying load qx/ℓ—are considered as shown in Table 1. For each loading case, linear and quadratic shape functions are used for the finite-element models. For the linear shape function, a two-element model is used, whereas for the quadratic shape function, a one-element model is used. For the equivalence study, design sensitivities of the nodal displacements are considered.

For the design velocity $V(x)$ to be used in the C-D method, three parameterization methods—linear, quadratic, and cubic polynomials—are used as shown in Fig. 1, where

$$\alpha_1 = \frac{4\epsilon_1 - \delta\ell}{\ell} \tag{5a}$$

$$\alpha_2 = \frac{2\delta\ell - 4\epsilon_1}{\ell^2} \tag{5b}$$

$$\beta_1 = \frac{18\epsilon_2 - 9\epsilon_3 + 2\delta\ell}{2\ell} \tag{5c}$$

$$\beta_2 = \frac{-45\epsilon_2 + 36\epsilon_3 - 9\delta\ell}{2\ell^2} \tag{5d}$$

$$\beta_3 = \frac{27\epsilon_2 - 27\epsilon_3 + 9\delta\ell}{2\ell^3} \tag{5e}$$

Note that for all three parameterizations of design velocity, the perturbation of the length of the truss is $\delta\ell$ at the tip. Moreover, for the quadratic and cubic design velocities, once ϵ_i, $i = 1,2,3$, are fixed, then the only shape design variable is the length ℓ. The movement of the finite-element grid points for the shape design change in the D-D method must be consistent with these parameterization methods. For the D-D method, the shape design variables are the positions b_1 and b_2 of the nodal points. If the present design is $b_1 = \ell/2$ and $b_2 = \ell$, then $V(\ell/2) \equiv \delta b_1 = \delta\ell/2$ and $V(\ell) \equiv \delta b_2 = \delta\ell$ for the linear velocity, $V(\ell/2) \equiv \delta b_1 = \epsilon_1$ and $V(\ell) \equiv \delta b_2 = \delta\ell$ for the quadratic velocity, and $V(\ell/2) \equiv \delta b_1 = (9\epsilon_2 + 9\epsilon_3 - \delta\ell)/16$ and $V(\ell) \equiv \delta b_2 = \delta\ell$ for the cubic velocity.

The results of the equivalence study are summarized in Table 1. The first case of study is the truss with the point load p at the tip. For this, the finite-element matrix equation, using the linear shape function, is

$$EA \begin{bmatrix} \dfrac{b_2}{b_1(b_2 - b_1)} & \dfrac{1}{b_1 - b_2} \\ \dfrac{1}{b_1 - b_2} & \dfrac{1}{b_2 - b_1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix} \tag{6}$$

which gives the solutions $z_1 \equiv z(\ell/2) = p\ell/2EA$ and $z_2 \equiv z(\ell) = p\ell/EA$ at the present design $b_1 = \ell/2$ and $b_2 = \ell$. Thus, $z(x) = px/EA$ which is the exact solution of the truss with the point load p. If the design sensitivities of z_1 and z_2 are desired, the adjoint equations are

$$\frac{2EA}{\ell} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7}$$

and

$$\frac{2EA}{\ell} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8}$$

with the adjoint solutions $\lambda^1(x) = x/EA$ for $0 \le x \le \ell/2$, $\lambda^1(x) = \ell/2EA$ for $\ell/2 \le x \le \ell$, and $\lambda^2(x) = x/EA$, respectively. These adjoint solutions are exact.

Using the D-D method, the design sensitivities for z_1 and z_2 are $z_1' = p\delta\ell/2EA$ and $z_2' = p\delta\ell/EA$, respectively, for the linear velocity. On the other hand, if the quadratic velocity is used, then $z_1' = p\epsilon_1/EA$ and $z_2' = p\delta\ell/EA$. Also for the cubic velocity, the D-D method yields $z_1' = p(9\epsilon_2 + 9\epsilon_3 - \delta\ell)/16EA$ and $z_2' = p\delta\ell/EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z_i' = \int_0^\ell EA z_x \lambda_x^i V_x \, dx, \qquad i = 1,2 \tag{9}$$

Using the finite-element analysis results and the linear velocity in Eq. (9), the C-D method gives $z_1' = p\delta\ell/2EA$ and $z_2' = p\delta\ell/EA$ which are the same as the results of the D-D method. Moreover, Eq. (9) yields $z_1' = p\epsilon_1/EA$ and $z_2' = p\delta\ell/EA$ for the quadratic velocity and $z_1' = p(9\epsilon_2 + 9\epsilon_3 - \delta\ell)/16EA$ and $z_2' = p\delta\ell/EA$ for the cubic velocity, which are the same as the results of the D-D method. Thus, the D-D and C-D methods are equivalent for the truss with the point load p when the linear shape function is used for the finite-element model for all parameterizations of velocity considered as indicated in the second column of Table 1. One point to emphasize in this case is that the original and adjoint responses of finite-element models are exact solutions of the truss. Note that the design sensitivity $z_2' = p\delta\ell/EA$ is independent of the parameterizations of velocity for the C-D method.

If the quadratic shape function is used, the same finite-element solution $z(x) = px/EA$ as in the linear shape function case is obtained. As mentioned before, this is the exact solution of the truss with the point load p. The adjoint equations are

$$\frac{EA}{3\ell} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

and

$$\frac{EA}{3\ell} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

with the adjoint solutions $\lambda^1(x) = (-3x^2/4\ell + 5x/4)/EA$ and $\lambda^2(x) = x/EA$, respectively. The adjoint solution $\lambda^2(x)$ is the same as in the linear shape function case, which is the exact solution. On the other hand, the adjoint solution $\lambda^1(x)$ is different from the linear shape function case and is not exact. Since $z(x)$ and $\lambda^2(x)$ are exact, it can be expected that the D-D and C-D methods give the same $z_2' = p\delta\ell/EA$ for all parameterizations of velocity as in the linear shape function case. It is interesting that even though $\lambda^1(x) = (-3x^2/4\ell + 5x/4)/EA$ is different from the linear shape function case, the D-D and C-D methods yield the same result for the design sensitivity of z_1 as in the linear shape function case for all parameterizations of velocity considered. Hence, it is concluded that the D-D and C-D methods are equivalent in the first case of study as indicated in Table 1.

The second case of study is the truss with the uniformly distributed load f along the truss. For this, the finite-element matrix equation, using the linear shape function, is the same as Eq. (6) except with a different load vector

$$F(b) = \left[\frac{fb_2}{2}, \frac{f(b_2 - b_1)}{2}\right]^T \tag{12}$$

which gives the solutions $z_1 = z(\ell/2) = 3f\ell^2/8EA$ and $z_2 = z(\ell) = f\ell^2/2EA$ at the present design $b_1 = \ell/2$ and $b_2 = \ell$. Thus, $z(x) = 3f\ell x/4EA$ for $0 \le x \le \ell/2$ and $z(x) = f\ell(x + \ell)/4EA$ for $\ell/2 \le x \le \ell$, which is not the exact solution of the truss with the uniformly distributed load f. If the design sensitivities of z_1 and z_2 are desired, the adjoint equations are given in Eqs. (7) and (8), with the solutions $\lambda^1(x) = x/EA$ for $0 \le x \le \ell/2$, $\lambda^1(x) = \ell/2EA$ for $\ell/2 \le x \le \ell$, and $\lambda^2(x) = x/EA$, respectively. As mentioned before, these adjoint solutions are exact.

Using the D-D method, the design sensitivities for z_1 and z_2 are $z_1' = 3f\ell\delta\ell/4EA$ and $z_2' = f\ell\delta\ell/EA$, respectively, for the linear velocity. On the other hand, if the quadratic velocity is used, then $z_2'=f\ell\delta\ell/EA$. Also for the cubic velocity, the D-D method yields $z_2'=f\ell\delta\ell/EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z_i' = \int_0^\ell (f\lambda^i + EAz_x\lambda_x^i)V_x\,dx, \qquad i = 1,2 \tag{13}$$

Using the finite-element analysis results and the linear velocity in Eq. (13), the C-D method gives $z_1' = 3f\ell\delta\ell/4EA$ and $z_2'=f\ell\delta\ell/EA$, which are the same as the results of the D-D method. However, Eq. (13) yields $z_2'=f\ell(13\delta\ell-2\epsilon_1)/12EA$ for the quadratic velocity and $z_2'=f\ell(35\delta\ell-3\epsilon_2-3\epsilon_3)/32EA$ for the cubic velocity, which are not the same as the results of the D-D method. Thus, the D-D and C-D methods are not equivalent for the truss with the uniformly distributed load f when the linear shape function is used for the finite-element model.

If the quadratic shape function is used, the finite-element matrix equation is

$$EA\begin{bmatrix} \dfrac{b_2^3}{3b_1^2(b_1-b_2)^2} & -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} \\ -\dfrac{b_2^2}{3b_1(b_1-b_2)^2} & \dfrac{4b_2^2-6b_1b_2+3b_1^2}{3b_2(b_1-b_2)^2}\end{bmatrix}\begin{bmatrix} z_1 \\ z_2\end{bmatrix} = \begin{bmatrix} \dfrac{fb_2^3}{6b_1(b_2-b_1)} \\ \dfrac{fb_2(2b_2-3b_1)}{6(b_2-b_1)}\end{bmatrix} \tag{14}$$

which gives the solutions $z_1=z(\ell/2)=3f\ell^2/8EA$ and $z_2=z(\ell)=f\ell^2/2EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus, $z(x)=fx(-x+2\ell)/2EA$, which is the exact solution of the truss with the uniformly distributed load f. The adjoint equations are Eqs. (10) and (11) with the solutions $\lambda^1(x)=(-3x^2/4\ell+5x/4)/EA$ and $\lambda^2(x)=x/EA$, respectively. As mentioned before, the adjoint solution $\lambda^1(x)$ is not exact, whereas $\lambda^2(x)$ is exact. The design sensitivity results of the D-D method are $z_1'=3f\ell\delta\ell/4EA$ and $z_2'=f\ell\delta\ell/EA$ for the linear velocity, $z_1'=f\ell(\delta\ell+\epsilon_1)/2EA$ and $z_2'=f\ell\delta\ell/EA$ for the quadratic velocity, and $z_1'=f\ell(15\delta\ell+9\epsilon_2+9\epsilon_3)/32EA$ for the cubic velocity. Now using the finite-element analysis results and the linear velocity in Eq. (13), the C-D method gives $z_1'=3f\ell\delta\ell/4EA$ and $z_2'=f\ell\delta\ell/EA$, which are the same as the results of the D-D method. Also, using the finite-element analysis results and the quadratic velocity in Eq. (13), the C-D method gives $z_1'=f\ell(\delta\ell+\epsilon_1)/2EA$ and $z_2'=f\ell\delta\ell/EA$. These are the same as the results of the D-D method. However, Eq. (13) yields $z_1'=f\ell(42\delta\ell+36\epsilon_2+9\epsilon_3)/80EA$ for the cubic velocity, which is different from the result of the D-D method. Hence, it can be concluded that the D-D and C-D methods are not equivalent in the second case of study.

Notice that the sensitivity results of the D-D method are the same as those of the C-D method up to the linear velocity when the linear shape function is used and up to the quadratic velocity when the quadratic shape function is used. Thus, the second case indicates that the D-D and C-D methods might be equivalent under an additional condition that the shape function used in the finite-element model is isoparametric with the discretization polynomial of the design velocity. However, this is not true as the results of the next case of study indicate.

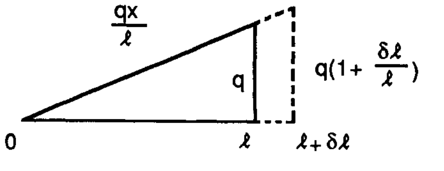

Fig. 2 Dependency of the external load on the shape design variable.

The last case of study is the truss with the linearly varying load qx/ℓ along the truss. Before carrying out the design sensitivity computation, dependency of the external load on the shape design has to be defined as shown in Fig. 2. That is, as the length of the truss changes, the external load will maintain the form of qx/ℓ. For this, the finite-element matrix equation, using the linear shape function, is the same as Eq. (6) except with a different load vector

$$F(b) = \left[\frac{q(b_1+b_2)}{6},\ \frac{q(b_2-b_1)(2b_2+b_1)}{6b_2}\right]^T \tag{15}$$

which gives the solutions $z_1=z(\ell/2)=11q\ell^2/48EA$ and $z_2=z(\ell)=q\ell^2/3EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus, $z(x)=11q\ell x/24EA$ for $0\le x\le \ell/2$ and $z(x)=q\ell(10x+11\ell)/48EA$ for $\ell/2\le x\le\ell$, which is not the exact solution of the truss with linearly varying load. If the design sensitivities of z_1 and z_2 are desired, the adjoint equations are given in Eqs. (7) and (8), with the solutions $\lambda^1(x)=x/EA$ for $0\le x\le\ell/2$, $\lambda^1(x)=\ell/2EA$ for $\ell/2\le x\le\ell$, and $\lambda^2(x)=x/EA$, respectively. As mentioned before, these adjoint solutions are exact.

Using the D-D method, the design sensitivities for z_1 and z_2 are $z_1'=11q\ell\delta\ell/16EA$ and $z_2'=q\ell\delta\ell/EA$, respectively, for the linear velocity. On the other hand, if the quadratic velocity is used, then $z_2'=q\ell\delta\ell/EA$. Also for the cubic velocity, the D-D method yields $z_2'=q\ell\delta\ell/EA$. Now, using the C-D method, the design sensitivity expression is obtained as

$$z_i' = \int_0^\ell\left[\left(\frac{q}{\ell}\right)\lambda^i V + \left(\frac{qx}{\ell}\right)\lambda^i V_x + EAz_x\lambda_x^i V_x\right]dx, \quad i = 1,2 \tag{16}$$

Using the finite-element analysis results and the linear velocity in Eq. (16), the C-D method gives $z_1' = 11q\ell\delta\ell/16EA$ and $z_2'=q\ell\delta\ell/EA$, which are the same as the results of the D-D method. However, Eq. (16) yields $z_2'=q\ell(25\delta\ell-2\epsilon_1)/24EA$ for the quadratic velocity and $z_2'=q\ell(347\delta\ell-21\epsilon_2+51\epsilon_3)/320EA$ for the cubic velocity, which are not the same as the results of the D-D method. Thus, the D-D and C-D methods are not equivalent for the truss with a linearly varying load when the linear shape function is used in the finite-element model.

If the quadratic shape function is used, the finite-element matrix equation is the same as Eq. (14) except with a different load vector

$$F(b) = \left[\frac{qb_2^3}{12b_1(b_2-b_1)},\ \frac{qb_2(3b_2-4b_1)}{12(b_2-b_1)}\right]^T \tag{17}$$

which gives the solutions $z_1=z(\ell/2)=11q\ell^2/48EA$ and $z_2=z(\ell)=q\ell^2/3EA$ at the present design $b_1=\ell/2$ and $b_2=\ell$. Thus, $z(x)=qx(-3x+7\ell)/12EA$, which is not the exact solution of the truss with the linearly varying load. The adjoint equations are Eqs. (10) and (11) with the solutions $\lambda^1(x)=(-3x^2/4\ell+5x/4)/EA$ and $\lambda^2(x)=x/EA$, respectively. As mentioned before, the adjoint solution $\lambda^1(x)$ is not exact, whereas $\lambda^2(x)$ is exact. The design sensitivity results of the D-D method are $z_1'=11q\ell\delta\ell/16EA$ and $z_2'=q\ell\delta\ell/EA$ for the linear velocity, $z_1'=q\ell(25\delta\ell+16\epsilon_1)/48EA$ for the quadratic velocity, and $z_2'=q\ell\delta\ell/EA$ for the cubic velocity. Now using the finite-element analysis results and the linear velocity in Eq. (16), the C-D method gives $z_1'=11q\ell\delta\ell/16EA$ and $z_2'=q\ell\delta\ell/EA$, which are the same as the results of the D-D method. However, using the finite-element analysis results and the quadratic velocity in Eq. (16), the C-D method gives $z_1'=q\ell(19\delta\ell+292\epsilon_1)/240EA$, whereas Eq. (16) yields $z_2'=q\ell(249\delta\ell+27\epsilon_2-27\epsilon_3)/240EA$ for the cubic velocity. These are not the same as the results of the D-D method. Hence, it can be concluded that the D-D and C-D methods are not equivalent in the last case of study. Notice that the sensitivity results of the D-D method are the same as those of the C-D method only for the linear velocity for all shape functions used. Thus, the D-D and C-D methods are not equivalent even under the additional condition that the shape function used in the finite-element model is isoparametric with the discretization polynomial of the design velocity.

Based on the equivalence study of truss problem, the D-D and C-D methods are possibly equivalent only for linear velocity. If this is the case, then both methods will give the exact design sensitivity information on the finite-element analysis results that may not be acceptable at all. This is the situation for the fillet problem in Ref. 21 that the design sensitivity results of the C-D method agree up to 5 to 6 digits with the finite difference, even though the finite-element model using a constant stress triangular element does not provide an accurate analysis result. On the other hand, limiting the design velocity field to linear functions will not be desirable at all since it cannot work well with quadratic isoparameteric finite elements. Moreover, when automatic regridding methods are employed for shape optimal design,[22,23] parameterizations of the design velocity field cannot be limited only to linear functions.

IV. Beam

In this section, an equivalence study has been carried out analytically and numerically using a simple cantilever beam with moment of inertia I and length ℓ. As with the truss problem, three lateral loading cases shown in the Table 2 are considered. For all loading cases, Hermite cubic shape functions are used for the finite-element model with one element. Also, for the design velocity $V(x)$, the same linear and quadratic parameterizations as in the truss problem are used. In addition to these, Hermitian parameterization of the velocity is used. That is, if the beam is fixed at $x=0$ and its length changes by $\delta\ell$ at $x=\ell$, and the slope of the velocity is zero at $x=0$ and θ at $x=\ell$, then

$$V_4(x) = \gamma_3 x^3 + \gamma_2 x^2 \tag{18}$$

where

$$\gamma_3 = \frac{\ell\theta - 2\delta\ell}{\ell^3} \tag{19a}$$

$$\gamma_2 = \frac{-\ell\theta + 3\delta\ell}{\ell^2} \tag{19b}$$

For the equivalence study, the design sensitivity of the tip displacement is considered. The results of the equivalence study are summarized in Table 2. The first case of study is the beam with the point load p at the tip. For this, the finite-element matrix equation is

$$\frac{EI}{b^3}\begin{bmatrix} 12 & -6b \\ -6b & 4b^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} \tag{20}$$

which gives tip displacement $z_1 = p\ell^3/3EI$ and tip rotation $z_2 = p\ell^2/2EI$ at the present design $b=\ell$. Thus, $z(x) = px^2(3\ell - x)/6EI$, which is the exact solution of the beam with point load. If the design sensitivity of z_1 is desired, the adjoint equation is

$$\frac{EI}{b^3}\begin{bmatrix} 12 & -6b \\ -6b & 4b^2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{21}$$

with the adjoint solution $\lambda(x) = x^2(3\ell - x)/6EI$, which is the exact solution. For the C-D method, the design sensitivity expression is obtained as

$$z_1' = \int_0^\ell EI\left[3z_{xx}\lambda_{xx}V_x + (z_x\lambda_{xx} + z_{xx}\lambda_x)V_{xx}\right] \mathrm{d}x \tag{22}$$

As in the truss problem, since the finite-element solutions of the original and adjoint systems are exact solutions, the D-D and C-D methods yield the same sensitivity result, $z_1' = p\ell^2\delta\ell/EI$, for all parameterizations of velocity considered. Thus, the D-D and C-D methods are equivalent for the beam

Table 2 Results of equivalence study of D-D and C-D methods for beam

	Loading condition		
Design velocity field	p	f	$qx/\ell \quad q$
Linear	Same	Same	Same
Quadratic	Same	Not	Not
Hermitian	Same	Not	Not

with point load as indicated in the Table 2. Also, since the finite-element solutions of the original and adjoint systems are exact solutions, the design sensitivity result $z_1' = p\ell^2\delta\ell/EI$ is independent of the parameterizations of velocity for the C-D method.

The second case of study is the beam with uniformly distributed load f along the beam. The finite-element matrix equation is the same as Eq. (20) except with a different load vector

$$F(b) = \left[\frac{fb}{2},\ -\frac{fb^2}{12}\right]^T \tag{23}$$

which gives the solutions $z_1 = f\ell^4/8EI$ and $z_2 = f\ell^3/6EI$ at the present design $b=\ell$. Thus, $z(x) = f\ell(5\ell x^2 - 2x^3)/24EI$, which is an approximate solution of the beam with the uniformly distributed load f. If the design sensitivity of z_1 is desired, the adjoint solution of Eq. (21) can be used.

Using the D-D method, the design sensitivity for z_1 is $z_1' = f\ell^3\delta\ell/2EI$ for all parameterizations of velocity. For the C-D method, the design sensitivity expression is obtained as

$$z_1' = \int_0^\ell \left\{EI\left[3z_{xx}\lambda_{xx}V_x + (z_x\lambda_{xx} + z_{xx}\lambda_x)V_{xx}\right] + f\lambda V_x\right\} \mathrm{d}x \tag{24}$$

Using the finite-element analysis results and the linear velocity in Eq. (24), the C-D method yields $z_1' = f\ell^3\delta\ell/2EI$, which is the same as the result of the D-D method. However, Eq. (24) yields $z_1' = f\ell^3(8\delta\ell - \epsilon_1)/15EI$ for the quadratic velocity and $z_1' = f\ell^3(64\delta\ell - \ell\theta)/120EI$ for the Hermitian velocity, which are not the same as the results of the D-D method. Thus, the D-D and C-D methods are not equivalent for the beam with uniformly distributed load as indicated in Table 2.

The last case of study is the beam with the linearly varying load qx/ℓ along the beam. The finite-element matrix equation is the same as Eq. (20) except with a different load vector

$$F(b) = \left[\frac{7qb}{20},\ -\frac{qb^2}{20}\right]^T \tag{25}$$

which gives the solutions $z_1 = 11q\ell^4/120EI$ and $x_2 = q\ell^3/8EI$ at the present design $b=\ell$. Thus, $z(x) = q\ell(18\ell x^2 - 7x^3)/120EI$, which is an approximate solution of the beam with linearly distributed load. For the design sensitivity of z_1, the adjoint solution of Eq. (21) can be used.

Using the D-D method, the design sensitivity for z_1 is $z_1' = 11q\ell^3\delta\ell/24EI$ for all parameterizations of velocity. For the C-D method, the design sensitivity expression is

$$z_1' = \int_0^\ell \left\{EI\left[3z_{xx}\lambda_{xx}V_x + (z_x\lambda_{xx} + z_{xx}\lambda_x)V_{xx}\right] + \left(\frac{q}{\ell}\right)\lambda V + \left(\frac{qx}{\ell}\right)\lambda V_x\right\} \mathrm{d}x \tag{26}$$

Using the finite-element analysis results in Eq. (26), the C-D method yields $z_1' = 11q\ell^3\delta\ell/24EI$ for the linear velocity, which is the same as the result of the D-D method. However, Eq. (26) yields $z_1' = q\ell^3(171\delta\ell - 12\epsilon_1)/360EI$ for the quadratic velocity and $z_1' = q\ell^3(1194\delta\ell - 9\ell\theta)/2520EI$ for the Hermitian velocity, which are not the same as the results of the D-D method. Thus, the D-D and C-D methods are not equivalent for the beam with linearly distributed load as indicated in Table 2. Based on the equivalence study of the beam problem, the D-D and C-D methods are possibly equivalent only for linear velocity.

Next, a numerical study is carried out for the C-D method using the cantilever beam with the uniformly distributed load to see the effect of accuracy of the finite-element analysis results on the accuracy of the design sensitivity information obtained. The finite-element models with 1, 2, and 20 elements are considered for numerical study. Node numbering for all finite-element models starts at the clamped end of the beam, and the node number of the free end of the beam is $(m+1)$ where m is the number of elements in the model. The beam is 60 in. long and has a uniform rectangular cross section of 0.5 in. high and 0.25 in. wide. Young's modulus, Poisson's ratio, and the uniformly distributed load are $E = 30 \times 10^6$ psi, $\nu = 0.3$, and $f = 0.5$ lb/in., respectively. Finite-element analysis is carried out using ANSYS finite-element STIF4. Three parameterizations of velocity with 1% perturbation of the length $\ell = 60$ in. of the beam are used for the numerical study as shown in Table 3.

Once the solutions of the original and adjoint structural system are obtained using ANSYS, the design sensitivity expression in Eq. (24) is numerically integrated using three-points Gauss quadrature. To check the accuracy of the design sensitivity obtained, the results are compared with the results obtained by finite difference as shown in Table 4. In Table 4, $z(\ell-\delta\ell)$ and $z(\ell+\delta\ell)$ are the displacements of selected nodal points for perturbed backward and forward designs, respectively, $\Delta z = z(\ell+\delta\ell) - z(\ell-\delta\ell)$ is the finite difference, and z' is the difference predicted by design sensitivity. The ratio of z' and $\Delta z \times 100$ can be used as a measure of accuracy of the design sensitivity.

Table 3 Parameterizations of velocity for numerical study of the C-D method

Case	Design velocity type	Parameter values
A	Linear	$\delta = 0.6$ in.
B	Quadratic	$\delta = 0.6$ in., $\epsilon_1 = 10$ in.
C	Hermitian	$\delta = 0.6$ in., $\theta = -0.3$

In Table 4, for all finite-element models, case A with linear velocity yields excellent agreement between the design sensitivity z' and the finite difference Δz. This confirms the results of the analytic study that the D-D and C-D methods may be equivalent for linear velocity. On the other hand, for the one-element model, the design sensitivity z' and the finite difference Δz do not agree at all for other parameterizations (cases B and C) of velocity as can be seen in Table 4a. For cases B and C, the agreements improve substantially for the two-element model, whereas for the 20-element model, agreements become excellent as shown in Table 4c. This confirms the fact that accurate design sensitivity information can be obtained as long as accurate finite-element analysis results are used for the C-D methods. This fact is not the case for the semianalytic method, as demonstrated by Barthelemy and Haftka[12] and later confirmed by Pedersen et al.[13] They found that the design sensitivity error of the semianalytic method is proportional to the square of the number of elements. This is completely opposite behavior from the C-D method since the design sensitivity error increases very rapidly as the finite-element analysis results of the original structure become more accurate.

As demonstrated in Table 4, an essential advantage that may accrue in the C-D method is associated with the ability to identify the effect of numerical error associated with the finite-element analysis results. That is, if disagreement arises between the design sensitivity of the C-D method and the finite difference, then error has crept into the finite-element approximation. If the D-D method is used, in which the structure is discretized and the design variables are imbedded into the stiffness matrix, then any error inherent in the finite-element model is consistently parameterized and will never be reported to the user. Therefore, precise design sensitivity coefficients of the matrix model of the structure are obtained without realizing that there may be substantial inherent error in the original model. On the other hand, the C-D method can be used to

Table 4 Comparison of design sensitivity of the C-D method

Case	Node no.	$z(\ell-\delta\ell)$	$z(\ell+\delta\ell)$	Δz	z'	$(z'/\Delta z \times 100)$%
			a) One-element model			
A	2	0.99593E+00	0.10789E+01	0.41476E−01	0.41471E−01	100.0
B	2	0.99593E+00	0.10789E+01	0.41476E−01	−0.47926E−01	−115.6
C	2	0.99593E+00	0.10789E+01	0.41476E−01	0.64970E−01	156.6
			b) Two-element model			
A	2	0.35273E+00	0.38210E+00	0.14689E−01	0.14688E−01	100.0
	3	0.99593E+00	0.10789E+01	0.41476E−01	0.41471E−01	100.0
B	2	0.17939E+00	0.59469E+00	0.20765E+00	0.20744E+00	99.9
	3	0.99593E+00	0.10789E+01	0.41476E−01	0.35880E−01	86.5
C	2	0.30947E+00	0.42955E+00	0.60040E−01	0.61126E−01	101.8
	3	0.99593E+00	0.10789E+01	0.41476E−01	0.42939E−01	103.5
			c) Twenty-element model			
A	2	0.48158E−02	0.52169E−02	0.20055E−03	0.20053E−03	100.0
	6	0.10504E+00	0.11379E+00	0.43744E−02	0.43739E−02	100.0
	11	0.35273E+00	0.38210E+00	0.14689E−01	0.14688E−01	100.0
	16	0.66525E+00	0.72066E+00	0.27704E−01	0.27701E−01	100.0
	21	0.99593E+00	0.10789E+01	0.41476E−01	0.41471E−01	100.0
B	2	0.70800E−03	0.13220E−01	0.62558E−02	0.62566E−02	100.0
	6	0.29727E−01	0.22933E+00	0.99800E−01	0.10127E+00	101.5
	11	0.17939E+00	0.59469E+00	0.20765E+00	0.21023E+00	101.2
	16	0.50771E+00	0.89205E+00	0.19217E+00	0.19269E+00	100.3
	21	0.99593E+00	0.10789E+01	0.41476E−01	0.41467E−01	100.0
C	2	0.47614E−02	0.52749E−02	0.25676E−03	0.25673E−03	100.0
	6	0.95061E−01	0.12480E+00	0.14869E−01	0.14863E−01	100.0
	11	0.30947E+00	0.42955E+00	0.60040E−01	0.60046E−01	100.0
	16	0.60858E+00	0.78134E+00	0.86383E−01	0.86384E−01	100.0
	21	0.99593E+00	0.10789E+01	0.41476E−01	0.41470E−01	100.0

obtain a warning that approximation error is creeping into the finite-element model.

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